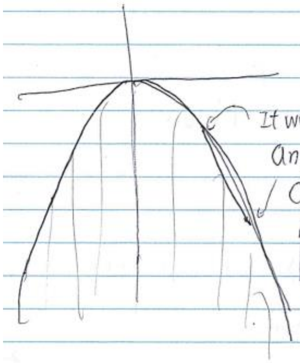
Math 3210 Tutorial 4

Example 1: Consider the set $y \leq -x^2$, try to show that any point on the closure is an extreme point

Again, it is easier to prove its contrapositive

Thinking:

We need to prove that if P is not an extreme point, then it is not on the boundary of $y=-x^2$.



If we chose two points that is in the set, any point in between will not be on the closure.

i.e in the interior.

For the point to be in the interior

$y<-x^2$

For the point to be on the closure

$y=-x^2$

An important inequality.

If $f(x)$ is convex 

~~$f(ax)$~~

$\lambda f(a_1) + (1-\lambda) f(a_2)$

$\geq f(\lambda a_1 + (1-\lambda) a_2)$.

$0 \leq \lambda \leq 1$

equality only occur at linear case

$f(x)$ is concave.

$\lambda f(a_1) + (1-\lambda) f(a_2)$

$\leq f(\lambda a_1 + (1-\lambda) a_2)$

$0 \leq \lambda \leq 1$

If we chose two points in the set.

case 1: both of the points are on the closure

$a_1 : \begin{pmatrix} x_1 \\ -x_1^2 \end{pmatrix} \quad a_2 : \begin{pmatrix} x_2 \\ -x_2^2 \end{pmatrix}$

$a_p = \begin{pmatrix} \lambda x_1 + (1-\lambda) x_2 \\ -[\lambda x_1^2 + (1-\lambda) x_2^2] \end{pmatrix} \begin{array}{l} x_p \\ y_p \end{array}$ Since $f(x) = -x^2$ is concave.

$\cancel{+(\lambda x_1^2) + [-(1-\lambda) x_2^2]}$

$= \begin{pmatrix} \lambda_1 x_1 + (1-\lambda) x_2 \\ [\lambda(-x_1^2) + (1-\lambda)(-x_2^2)] \end{pmatrix} \begin{array}{l} x_p \\ y_p \end{array}$ $\quad \lambda(-x_1^2) + (1-\lambda)(-x_2^2) \leq -(\lambda x_1 + (1-\lambda) x_2)^2$

$\cancel{\Rightarrow y_p \leq -y_p^2}$

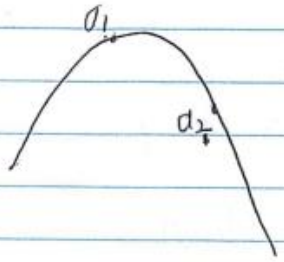
y_p

$\Rightarrow y_p < -x_p^2$

i.e it is in the interior.

Cases two : 1 of them is not on the closure.

$a_1 : \begin{pmatrix} x_1 \\ -x_1^2 \end{pmatrix}$ $a_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ ~~where $y_2 \neq <$~~



~~wh.~~

where $y_2 < -x_2^2$

$\cancel{a_p = \begin{pmatrix} \lambda x_1 + (1-\lambda) x_2 \\ \lambda(-x_1^2) + (1-\lambda)(-x_2^2) \end{pmatrix}}$ $a_p = \begin{pmatrix} \lambda x_1 + (1-\lambda) x_2 \\ \lambda(-x_1^2) + (1-\lambda) y_2 \end{pmatrix}$

note : $\lambda(-x_1^2) + (1-\lambda) y_2$
$< \lambda(-x_1^2) + (1-\lambda)(-x_2^2)$
$< -(\lambda x_1 + (1-\lambda) x_2)^2$.

Case three : Same as above

Potential method in finding/ improving basic solution

Step 1: Solve for a feasible solution $A_1 x_1 + \cdots + A_n x_n = B$

Step 2: Solve for no trivial solution among $\alpha_1 A_1 + \cdots + \alpha_n A_n = 0$

Step 3: combine the two equation to cancel out a particular variables

Example 2: Try to find a basic solution for the following system

Finding basic solution ~~for~~ through rearranging.
Consider :

$A = \begin{pmatrix} 1 & 5 & 3 \\ 4 & 9 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 14 \\ 23 \end{pmatrix}$

$\quad\ A_1 \quad A_2 \quad A_3$

~~$X_r =$~~

1^{st} find a feasible set :

$AX = B$ $X = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ $A_1 + 2A_2 + A_3 = B$. — (1)

2^{nd} try to find a non trivial solution for

$a_1 A_1 + a_2 A_2 + a_3 A_3 = 0$.
$2A_1 - A_2 + A_3 = 0$. — (2)

Want to find the solution that $X_2 = 0$

$$X_1 A_1 + X_2 A_2 = B.$$

From (2) $\quad 2A_1 + A_3 = A_2$

$$A_1 + 2(2A_1 + A_3) + A_3 = B$$

$$5A_1 + 3A_3 = B \qquad \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix} \text{ is a solu}$$

$\begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix}$ is a feasible basic solution

Next: try to set $X_3 = 0$

(2) $\rightarrow A_3 = A_2 - 2A_1$

(1) $\rightarrow A_1 + 2A_2 + (A_2 - 2A_1) = B.$

$A_2 - A_1 = B \rightarrow$ Not feasible.

More on basic solution and extreme points

Basic criteria for finding a basic solution for it to correspond to an extreme point:

Some Big Picture + Example. (平行時空)

Turning F.S $\rightarrow$ Basic Feasible Solution

$$\left(\boxed{A_1} \boxed{A_2} \cdots \boxed{A_n} \right) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = B \longrightarrow \left(\underset{\text{invertible}}{\boxed{A_1 \cdots A_m}} \cdots \right) \begin{pmatrix} x_1 \\ \vdots \\ x_m \\ 0 \\ 0 \end{pmatrix} = B. \quad B = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

Example 3:

Step 1: Find a feasible set of solution

$$\left(\boxed{A_1} \cdots \boxed{A_p}\ \square \cdots \square\right)\begin{pmatrix} x_1 \\ \vdots \\ x_p \\ 0 \\ 0 \end{pmatrix} = B$$

after rearrangement

$\boxed{\text{remark}}$

$$A = \begin{pmatrix} 2 & 7 & 3 \\ 5 & 11 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 15 \\ 18 \end{pmatrix}$$

$$X = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \quad x_1 = 1 \quad x_2 = 1 \quad x_3 = 2$$

Step 2: ~~Find~~ Assume $A_1, \cdots, A_p$ are L.D.

find a non trivial solution for
$\varphi_1 A_1 + \varphi_2 A_2 + \cdots + \varphi_p A_p = 0$

$$\varphi_1\begin{pmatrix} 2 \\ 5 \end{pmatrix} + \varphi_2\begin{pmatrix} 7 \\ 11 \end{pmatrix} + \varphi_3\begin{pmatrix} 3 \\ 1 \end{pmatrix} = 0$$

$$\varphi_1 = 2 \quad \varphi_2 = -1 \quad \varphi_3 = 1$$

Step 3: Reduce $A_1, \cdots, A_p$

$$\varphi_1 \vec{A}_1 + \varphi_2 \vec{A}_2 + \cdots + \varphi_p \vec{A}_p = 0$$

~~$\varphi_r \vec{A}_r = \varphi_1 \vec{A}_1 + \cdots + \cdots$~~

$$\varphi_r \vec{A}_r = -\varphi_1 \vec{A}_1 - \cdots - \varphi_{r-1}\vec{A}_{r-1} - \varphi_{r+1}\vec{A}_{r+1} - \cdots - \varphi_p \vec{A}_p$$

$$\vec{A}_r = -\frac{\varphi_1}{\varphi_r}\vec{A}_1 - \cdots - \frac{\varphi_p}{\varphi_r}\vec{A}_p \quad \text{— (3)}$$

out r.

$\vec{A}_1 x_1 + \cdots + \vec{A}_r x_r + \cdots \vec{A}_p x_p = \beta$. — (1)

(3) into (1).

$\vec{A}_1 x_1 + \cdots + \vec{A}_{r-1} x_{r-1} + (\underbrace{-\frac{\varphi_1}{\varphi_r}\vec{A}_1 \cdots - \frac{\varphi_p}{\varphi_r}\vec{A}_p}_{\text{out } r}) x_r$

$+ \cdots + \vec{A}_p x_p = \beta$

$\vec{A}_1 \left(x_1 - \frac{\varphi_1 x_r}{\varphi_r}\right) + \vec{A}_2 \left(x_2 - \varphi_1 \frac{x_r}{\varphi_r}\right) +$

$\cdots + \vec{A}_p \left(x_p - \frac{\varphi_p x_r}{\varphi_r}\right) = \beta$

$$\begin{pmatrix} x_1 - \varphi_1 \frac{x_r}{\varphi_r} \\ x_2 - \varphi_2 \frac{x_r}{\varphi_r} \\ x_{r-1} - \varphi_{r-1} \frac{x_r}{\varphi_r} \\ x_{r+1} - \varphi_{r+1} \frac{x_r}{\varphi_r} \\ x_i \\ x_p - \varphi_p \frac{x_r}{\varphi_r} \end{pmatrix}$$

find the r s.t. $\frac{x_r}{\varphi_r} > 0$ and minimised.

$\frac{x_1}{\varphi_1} = \frac{1}{2} \quad \frac{x_2}{\varphi_2} = -1 \quad \frac{x_3}{\varphi_3} = 2$.

reduce x_1.

Step 4:

$$A_1 + A_2 + 2A_3 = B$$

$$2A_1 - A_2 + A_3 = 0$$

$$A_1 = \frac{1}{2}(A_2 - A_3).$$

$$\frac{1}{2}(A_2 - A_3) + A_2 + 2A_3 = B.$$

$$\frac{3}{2}A_2 + \frac{3}{2}A_3 = B.$$

Basic $\begin{pmatrix} 0 \\ \frac{3}{2} \\ \frac{3}{2} \end{pmatrix}$

Step 5:

Check whether (A_2, A_3) are linearly independent

Some important reminder:

Example 4:

Max: $3x_1 + 4x_2 - 2x_3 = 2$

Subject to:

$$x_1 + 3x_2 + x_3 + s_1 = 8$$
$$2x_1 + 10x_2 + 4x_3 + s_2 = 26$$
$$4x_1 + 8x_2 + 2x_3 + s_3 = 22$$

Try to from a basic solution by setting. $s_1 = s_2 = s_3 = 0$

$$\left(\begin{array}{ccc|c} 1 & 3 & 1 & 8 \\ 2 & 10 & 4 & 26 \\ 4 & 8 & 2 & 22 \end{array}\right) \qquad B x = \vec{b}$$

B $\vec{b}$

$$X = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \qquad X = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \text{feasible}$$

$\vec{A}_1 x_1 + \vec{A}_2$ $\quad \vec{A}_1 + 2\vec{A}_2 + \vec{A}_3 = \vec{b}$ — (1)

and $\vec{A}_1 - \vec{A}_2 + 2\vec{A}_3 = 0$ — (2)

$\boxed{x_1 = 1}$ $x_2 = 2$ $x_3 = 3$

$\boxed{\varphi_1 = 1}$ $\varphi_2 = -1$ $\varphi_3 = 2$.

$\vec{A}_1 = \vec{A}_2 - 2\vec{A}_3$ — (3)

$3\vec{A}_2 - \vec{A}_3 = \vec{b} \rightarrow$ Not feasible.

try ~~A~~ kick out x_2

$$\vec{A_2} = \vec{A_1} + 2\vec{A_3}$$

$$3\vec{A_1} + 3\vec{A_3} = b.$$

rearrang

$$\left(A_1 A_3 \boxed{} \right) \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} = b$$

$$\underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 2 & 4 & 0 \\ 4 & 2 & 1 \end{pmatrix}}_{B'} \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} = \vec{b}$$

→ degenerating.

$$\begin{pmatrix} 3 \\ 0 \\ 3 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

→ is a degenerating basic solution.